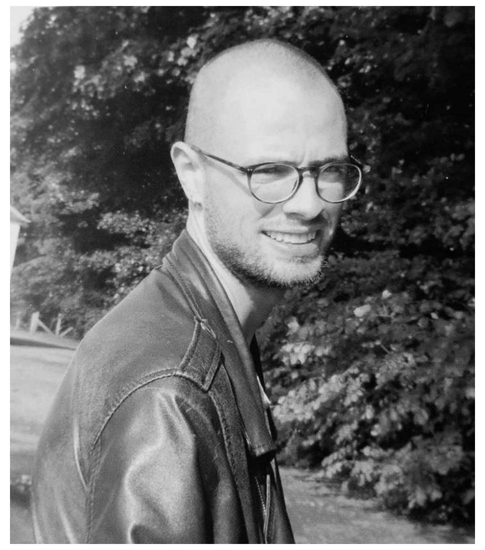
Can effective descriptions of bosonic systems be considered complete?

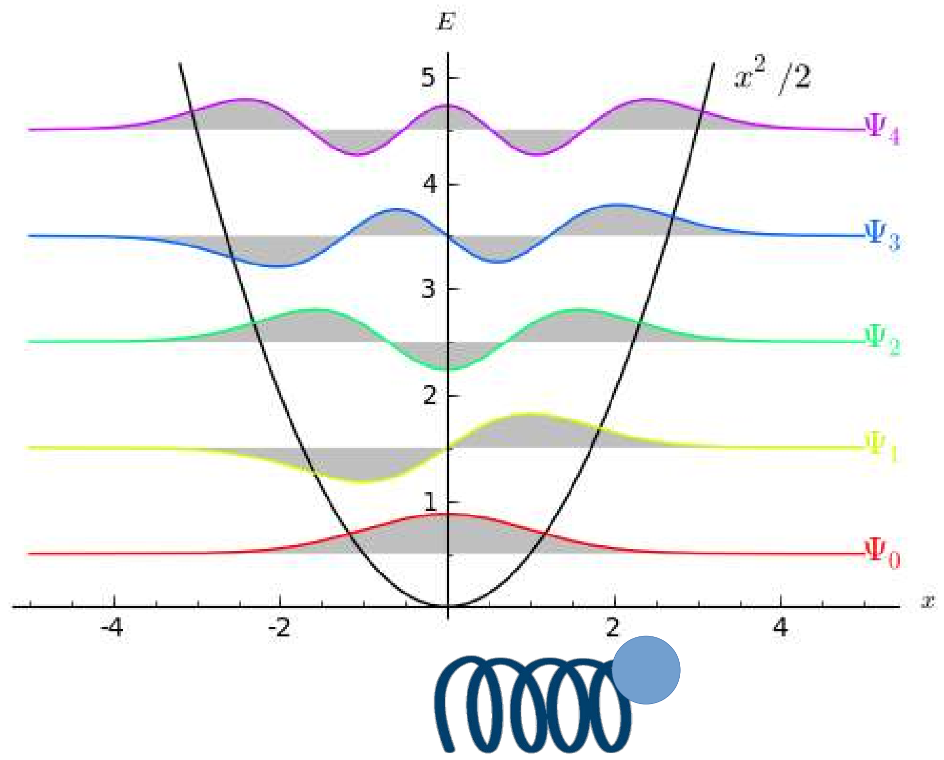
Francesco Arzani, Robert I. Booth, Ulysse Chabaud



arXiv:2501.13857

Context and motivation

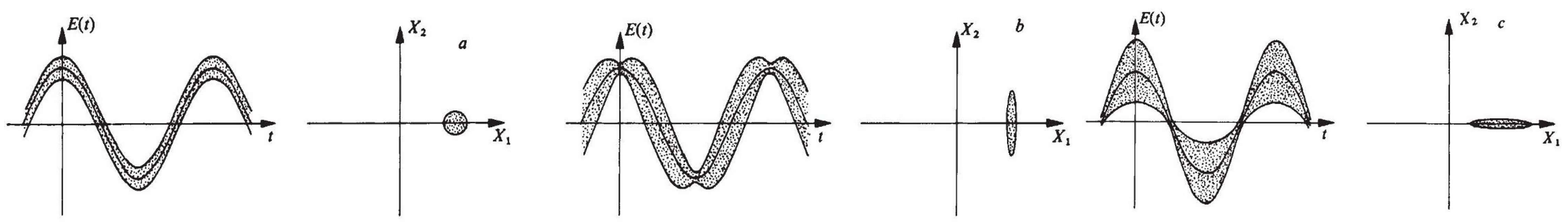
Bosonic systems



Bosonic systems appear in many natural and engineered settings
e.g. photonics, circuit QED, trapped ions, cold atoms, ...

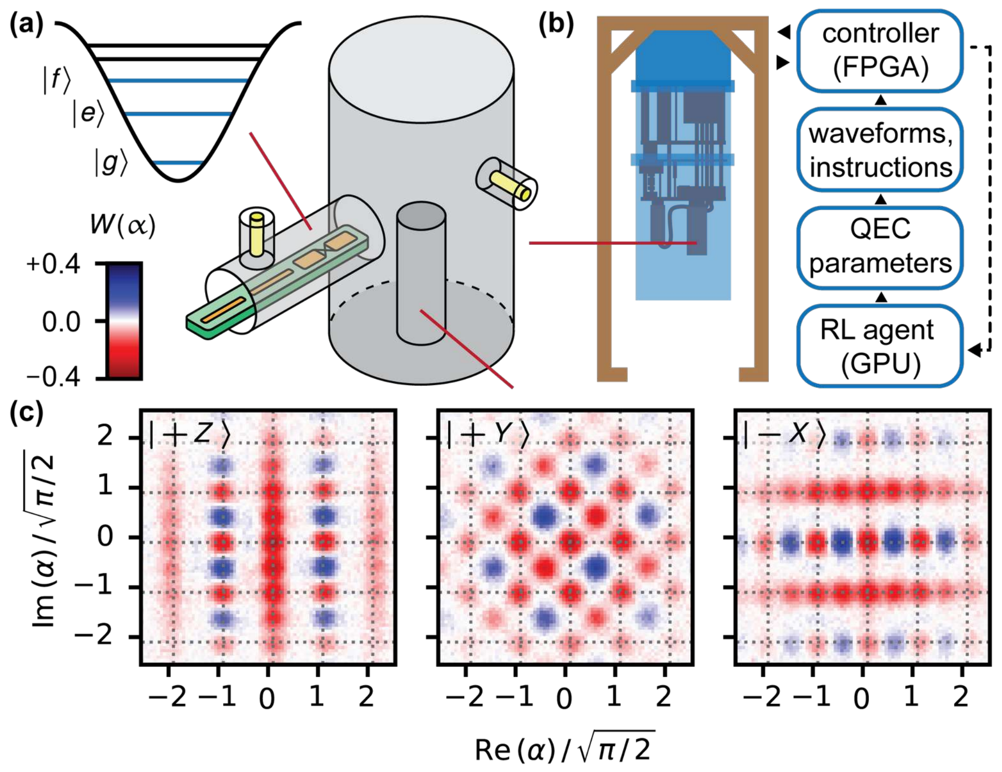
They are described by **infinite dimensional Hilbert spaces**.
This gives rise to unique phenomena (and opportunities):

Squeezed states of light



[Walls, Nature 1983]

Bosonic error correction



[Sivak et al, Nature 2023]

Also: gives rise to **mathematical complications** (unbounded operators, continuous spectra, ...)

Often, simplified, **effective descriptions** are used.
This work answers the fundamental question :

Can we trust such effective descriptions to capture physical effects?

Yes!

Effective descriptions of bosonic systems

Effective state space

- Truncating the infinite-dimensional Hilbert space to a finite *effective dimension*
- Based for instance on energy bounds
- Necessary to simulate bosonic systems on classical and DV quantum computers

Effective dynamics

- Restricting to specific families of *effective Hamiltonians*
- Polynomial Hamiltonians are ubiquitous in the physics of bosons

$$\hat{H} = \mathrm{poly}(\hat{q}_1, \hat{p}_1, \ldots, \hat{q}_m, \hat{p}_m)$$

- Used to define CV universality

Effective descriptions of bosonic systems

***what could go wrong*?**

Effective state space

- Bosonic CCR are impossible to satisfy in finite dimensions (even approximately)

$$[\hat{q}, \hat{p}] = i\hat{I}$$

- The set of quantum correlations differ in the finite and infinite-dimensional cases

[Coladangelo & Stark, Nat. Comm. 2020]

Effective dynamics

- Not all bosonic systems have native polynomial Hamiltonians

$$\cos \hat{q} \neq \sum_k \frac{\hat{q}^{2k}}{(2k)!}$$

- Polynomial Hamiltonians could generate a restricted set of unitary evolutions: is CV universality even well-defined?

Continuous-variable quantum computing

$$\left\{ e^{ir\hat{q}}, e^{is\hat{q}^2}, e^{it\hat{q}^3}, e^{iu\hat{q}_j\hat{q}_k}, e^{iv\left(\hat{q}^2+\hat{p}^2\right)} \right\}$$

$$e^{iA\delta t}e^{iB\delta t}e^{-iA\delta t}e^{-iB\delta t} = e^{(AB-BA)\delta t^2} + \boxed{O(\delta t^3)}, \qquad (1)$$

Potentially unbounded operator

Natural definition but hard to answer questions such as

"How many qubits are needed to simulate the dynamics of a system of bosons to a given accuracy?"
(quantum simulation)

"Are different models of bosonic quantum computers equally powerful?"
(quantum complexity theory)

"Can a universal bosonic computer prepare an arbitrary quantum state?"
(quantum control)

All answered by our work!

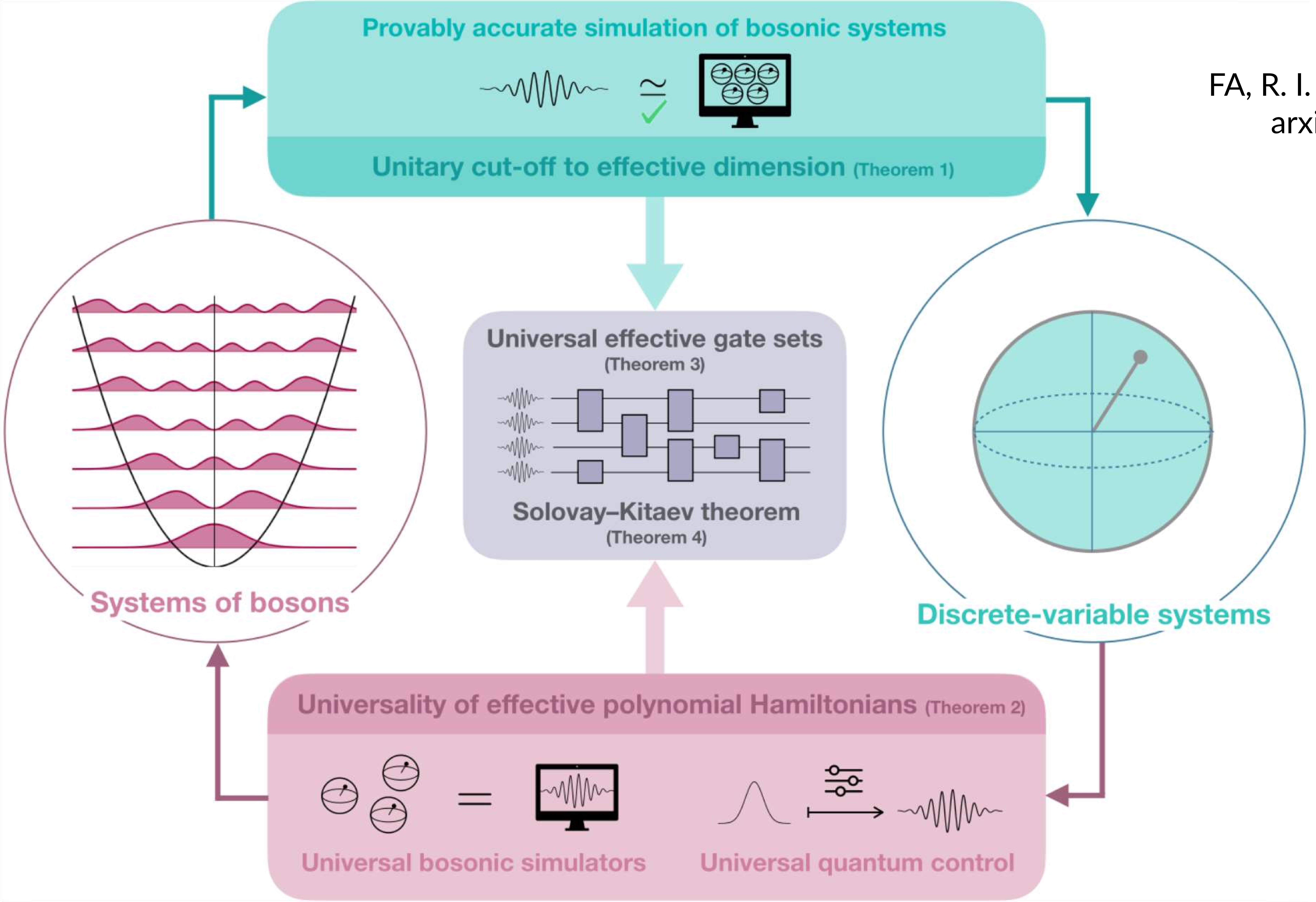
Discrete vs continuous-variable quantum computing

	DV	CV
Hilbert space	Finite-dimensional	Infinite-dimensional
Operators (observables)	Bounded (spin, polarisation, ...)	Bounded and unbounded (position, momentum, ...) $[\hat{q}, \hat{p}] = i\hat{I}$
Universal gate set	Can approximate any unitary evolution	Can approximate any unitary evolution *generated by polynomial Hamiltonians*
Solovay-Kitaev theorem	✓	✗

$\overset{?}{>}$

the “CV universality problem”

Results



FA, R. I. Booth, U. Chabaud
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Unitary truncation

Theorem 1
(unitary truncation)

Any infinite-dimensional *physical* unitary evolution can be rigorously approximated to arbitrary precision by a sequence of finite-dimensional ones

[ABC, arXiv:2501.13857]

Unitary truncation

Hilbert space $\mathcal{H} = \mathrm{span}\{|n\rangle\}_{n\in\mathbb{N}}$

Truncated Hilbert space $\mathcal{H}_N = \mathrm{span}\{|n\rangle\}_{0\leq n\leq N}$

Number operator $\hat{n}|n\rangle = n|n\rangle$

Energy = average particle number

Physical unitary operators: they map states of finite energy to states of finite energy

$$E_{\hat{U}}(N) := \sup_{|\psi\rangle\in\mathcal{H}_N} \langle\psi|\hat{U}^\dagger\hat{n}\hat{U}|\psi\rangle < +\infty$$

The right topology in infinite-dimensions: energy-constrained diamond norms

$$\|\mathcal{E}\|_\diamond^E := \sup_{\mathrm{Tr}[\rho(\hat{n}\otimes\hat{I})]\leq E} \|(\mathcal{E}\otimes\mathrm{id})\rho\|_1$$

[ABC, arXiv:2501.13857]

Unitary truncation

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$$\|\mathcal{E}\|_\diamond^E := \sup_{\mathrm{Tr}[\rho(\hat{n}\otimes\hat{I})]\leq E} \|(\mathcal{E}\otimes\mathrm{id})\rho\|_1$$

$$\hat{V}_d \oplus \hat{V} = \begin{pmatrix} V_d & (0) \\ (0) & V \end{pmatrix}$$

Theorem 1 (unitary truncation)

Any infinite-dimensional *physical* unitary evolution can be rigorously approximated to arbitrary precision by a sequence of finite-dimensional ones

$$d = \mathcal{O}\left(\frac{E}{\epsilon^4} E_{\hat{U}}\left(\frac{64E}{\epsilon^2}\right)\right)$$

$$\Downarrow$$

$$\exists \hat{V}_d \in \mathcal{U}(\mathcal{H}_d), \forall \hat{V}, \|\hat{U} - \hat{V}_d \oplus \hat{V}\|_\diamond^E \leq \epsilon$$

- Constructive proof: explicit finite-dimensional approximation (truncation + unitary rounding)
- Rigorous upper bound on the effective dimension necessary to simulate bosonic dynamics

[ABC, arXiv:2501.13857]

DV universality of polynomial Hamiltonians

Theorem 2
(DV universality of polynomial Hamiltonians)

Any finite-dimensional unitary evolution can be generated exactly by a polynomial Hamiltonian

[ABC, arXiv:2501.13857]

DV universality of polynomial Hamiltonians

$$\mathcal{H} = \mathrm{span}\{|n\rangle\}_{n\in\mathbb{N}}$$

$$\mathcal{H}_N = \mathrm{span}\{|n\rangle\}_{0\leq n\leq N}$$

$$\hat{A} \oplus \hat{A}' = \begin{pmatrix} A & (0) \\ (0) & A' \end{pmatrix}$$

- Constructive proof: explicit polynomial Hamiltonian (interpolation polynomials)

$$P_{|n\rangle\langle n|}(\hat{n}) = \prod_{\substack{k=0 \\ k\neq n}}^{N} \frac{\hat{n}-k}{n-k}$$

$$P_{|n\rangle\langle n+k|}(\hat{q},\hat{p}) = \sqrt{\frac{n!}{(n+k)!}} P_{|n\rangle\langle n|}(\hat{n})\hat{a}^k$$

Theorem 2
(DV universality of polynomial Hamiltonians)

Any finite-dimensional unitary evolution can be generated exactly by a polynomial Hamiltonian

$\hat{A}$ Hermitian operator on $\mathcal{H}_d$

$$\exists P_{\hat{A}} \in \mathbb{C}_{3d}[X,Y], \exists \hat{A}',\ P_{\hat{A}}(\hat{q},\hat{p}) = \hat{A} \oplus \hat{A}'$$

and

$$e^{iP_{\hat{A}}(\hat{q},\hat{p})} = e^{i\hat{A}} \text{ on } \mathcal{H}_d$$

[ABC, arXiv:2501.13857]

DV universality of polynomial Hamiltonians

$$\mathcal{H} = \mathrm{span}\{|n\rangle\}_{n\in\mathbb{N}}$$

$$\mathcal{H}_N = \mathrm{span}\{|n\rangle\}_{0\leq n\leq N}$$

$$\hat{A}\oplus\hat{A}' = \begin{pmatrix} A & (0) \\ (0) & A' \end{pmatrix}$$

- Constructive proof: explicit polynomial Hamiltonian (interpolation polynomials)
- Universal bosonic simulators

$$\hat{A} \mapsto P_{\hat{A}}(\hat{q},\hat{p})$$

- Universal controllability/state engineering

$$e^{i\hat{P}_\psi(\hat{q},\hat{p})}|0\rangle \simeq |\psi\rangle$$

Theorem 2
(DV universality of polynomial Hamiltonians)

Any finite-dimensional unitary evolution can be generated exactly by a polynomial Hamiltonian

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$\hat{A}$ Hermitian operator on $\mathcal{H}_d$

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and

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[ABC, arXiv:2501.13857]

Combining the technical theorems

Theorem 1
(unitary truncation)

Any infinite-dimensional *physical* unitary evolution can be rigorously approximated to arbitrary precision by a sequence of finite-dimensional ones

$+$

Theorem 2
(DV universality of polynomial Hamiltonians)

Any finite-dimensional unitary evolution can be generated exactly by a polynomial Hamiltonian

[ABC, arXiv:2501.13857]

CV universality of polynomial Hamiltonians

Theorem 1
(unitary truncation)

Any infinite-dimensional *physical* unitary evolution can be rigorously approximated to arbitrary precision by a sequence of finite-dimensional ones

$+$

Theorem 2
(DV universality of polynomial Hamiltonians)

Any finite-dimensional unitary evolution can be generated exactly by a polynomial Hamiltonian

$\Rightarrow$

Theorem 3
(CV universality of polynomial Hamiltonians)

Any infinite-dimensional *physical* unitary evolution can be rigorously approximated to arbitrary precision by a sequence of unitary gates generated by polynomial Hamiltonians

[ABC, arXiv:2501.13857]

CV universality of polynomial Hamiltonians

Proof sketch

Infinite-dimensional physical unitary evolution

$$E_{\hat{U}}(N) := \sup_{|\psi\rangle \in \mathcal{H}_N} \langle\psi|\hat{U}^\dagger \hat{n} \hat{U}|\psi\rangle < +\infty$$

$\wr\wr$ Unitary truncation (Th 1)

Finite-dimensional unitary evolution

$||$ DV universality of polynomial Hamiltonians (Th 2)

Sequence of infinite-dimensional unitary gates generated by polynomial Hamiltonians

Theorem 3
(CV universality of polynomial Hamiltonians)

Any infinite-dimensional *physical* unitary evolution can be rigorously approximated to arbitrary precision by a sequence of unitary gates generated by polynomial Hamiltonians

[ABC, arXiv:2501.13857]

CV universality of polynomial Hamiltonians

Resolves the CV universality problem:

Unitary gates generated by polynomial Hamiltonians form truly universal gate sets!

Solovay-Kitaev theorem?

Theorem 3
(CV universality of polynomial Hamiltonians)

Any infinite-dimensional *physical* unitary evolution can be rigorously approximated to arbitrary precision by a sequence of unitary gates generated by polynomial Hamiltonians

[ABC, arXiv:2501.13857]

DV Solovay-Kitaev theorem

Theorem
(DV Solovay-Kitaev)

Any (finite-dimensional) unitary evolution can be rigorously approximated to arbitrary precision by a ***short*** sequence of unitary gates from a universal gate set

Ensures that computational power is independent of the choice of universal gate set*

CV Solovay-Kitaev theorem

Theorem 4
(CV Solovay-Kitaev)

Any infinite-dimensional *physical* unitary evolution can be rigorously approximated to arbitrary precision by a ***short*** sequence of unitary gates generated by polynomial Hamiltonians

[ABC, arXiv:2501.13857]

CV Solovay-Kitaev theorem

Infinite-dimensional physical unitary evolution

$\wr\wr$ Unitary truncation (Th 1)

Finite-dimensional unitary evolution

$\wr\wr$ DV Solovay-Kitaev theorem

Short sequence of finite-dimensional unitary gates

$||$ DV universality of polynomial Hamiltonians (Th 2)

Short sequence of infinite-dimensional unitary gates generated by polynomial Hamiltonians

Theorem 4
(CV Solovay-Kitaev)

Any infinite-dimensional *physical* unitary evolution can be rigorously approximated to arbitrary precision by a ***short*** sequence of unitary gates generated by polynomial Hamiltonians

→ any choice of universal gate sets over $\mathcal{H}_d$ for all d gives rise to a CV universal set of unitary gates generated by polynomial Hamiltonians, satisfying the CV Solovay-Kitaev theorem

[ABC, arXiv:2501.13857]

In conclusion…

Summary

Effective descriptions of bosonic systems can be considered complete!

Unitary truncation

- Quantum features requiring infinite dimensions are not physical
- Rigorous upper bound on the effective dimension necessary to simulate bosonic dynamics
- Constructive proof: explicit finite-dimensional approximation

DV universality of polynomial Hamiltonians

- Universal controllability of polynomial Hamiltonians
- Constructive proof: universal state engineering and universal bosonic simulators

CV universality of polynomial Hamiltonians

- Solving the CV universality problem: polynomial Hamiltonians form truly universal gate sets!
- A CV Solovay-Kitaev theorem for a class of polynomial Hamiltonians

[ABC, arXiv:2501.13857]

Open questions

- Truncation theorem for non-unitary dynamics?
- Universality of hybrid boson-fermion polynomial Hamiltonians?
- Formal proof that the standard CV gate set $\{e^{i\frac{\pi}{2}(\hat{q}^2+\hat{p}^2)}, e^{ir\hat{q}_1\hat{q}_2}, e^{i\gamma\hat{q}^3}, e^{it\hat{q}}\}$ is universal is missing
  In general: can we extend universality results for any *finite set* of hamiltonians?
- More general / more efficient CV Solovay-Kitaev theorem?
- Experimental demonstration of universal bosonic quantum simulators?

[ABC, arXiv:2501.13857]